Symmetry and Representation Theory of Lie Groups and Lie Algebras

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Abstract

Representations of Lie groups are abstractions of continuous symmetries of linear spaces. By considering their differentials (linear approximations), we obtain representations of Lie algebras. These are regarded as generalizations of the Fourier analysis and important tools not only in mathematics but also in physics. In this article, I present fundamental and important examples on representations of Lie groups and Lie algebras and present some of my latest results.

Keywords: fundamental mathematics, Lie groups, Lie algebras

1. Introduction

Lie groups are abstractions of continuous symmetries of spaces, and linear symmetries are abstracted by representations of Lie groups [1]. These are useful for analyzing functions on spaces with symmetries. However, Lie groups are non-linear objects, and it is not easy to treat their representations directly. To overcome this non-linearity, it is useful to consider the representations of Lie algebras by taking the differentials (linear approximations) of representations of Lie groups. These differentials preserve much information on the original representations and are easier to treat.

2. Representations of Lie groups

First, a Lie group is defined as a subset of the set of all $n \times n$ invertible matrices with complex entries (which is denoted as $GL(n, \mathbb{C})$ and called the general linear group), closed by the products, the inverses, and taking the limits*1. For example, $GL(n, \mathbb{C})$,

$GL(n, \mathbb{R}) =$
$\{n \times n \text{ invertible matrices with real entries}\}$,
$SL(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) \mid \det(g) = 1\}$,
$O(n) = \{g \in GL(n, \mathbb{R}) \mid g^t g = I_n\}$, and

$U(n) = \{g \in GL(n, \mathbb{C}) \mid g^t \bar{g} = I_n\}$

are typical examples of Lie groups (where I_n is the identity matrix). Next, let G be a Lie group, and X be a space such that "convergence can be defined" (i.e., a topological space). If a transformation $\tau(g)$: $X \to X$ is given for each element $g \in G$ and if it satisfies the associative law and the continuity in a suitable sense, we say that G acts on X. For example, rotations of the unit disk (the disk of radius 1) around the origin are regarded as the action of the Lie group $U(1)$. Similarly, conformal transformations of the unit disk, i.e., transformations that preserve angles of two intersecting curves, are almost regarded as the action of the Lie group $SU(1, 1)$ (**Fig. 1**). These examples show that an action of G on X controls the symmetries of X.

When the space $X = V$ on which G acts is a linear space and $\tau(g)$ is a linear map on V (i.e., it preserves additions and scalar multiplications), (τ, V) is called a representation of G. For example, if G acts on X, then G acts automatically on the space of functions on X (e.g., $V = L^2(X) = \{f\colon X \to \mathbb{C} \mid \int_X |f(x)|^2 dx < \infty\}$: the

*1 More generally, Lie groups are defined as sets equipped with group and manifold structures such that the group operations are differentiable. Groups that are not realizable as closed subgroups of matrices but locally isomorphic to those are also called Lie groups.

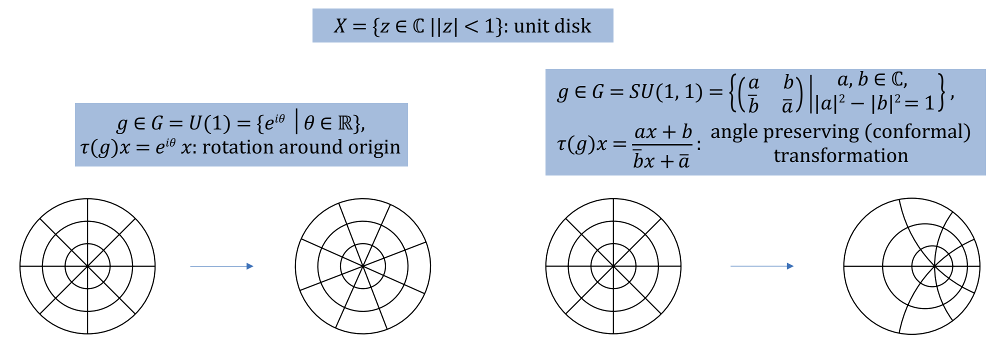


Fig. 1. Actions of $U(1)$, $SU(1, 1)$ on unit disk.

space of square integrable functions). This action on $L^2(X)$ is linear and becomes a representation of G. Such representation is in general infinite-dimensional and looks difficult, but in many cases, it consists of a sum of simpler representations. Therefore, to understand function spaces, it is important to understand simpler representations in detail.

The most elementary example of representations of a Lie group $G \subset GL(n, \mathbb{C})$ is $V := \mathbb{C}^n$ (the space of column vectors), with $\tau(g)$ defined by the product of matrices

$$\tau(g)\colon \mathbb{C}^n \to \mathbb{C}^n, \quad \tau(g)v := gv$$

for each $g \in G$. As more non-trivial examples, for a non-negative integer k, the action of the Lie group $G = U(n)$ on the linear space

$$V = \mathcal{P}_k(\mathbb{C}^n) := \{f(x) = f(x_1, \ldots, x_n)\text{: polynomial of } n \text{ variables} \mid f(tx) = t^k f(x)\ (t \in \mathbb{C})\}$$

(the space of homogeneous polynomials of n variables, degree k), with $\tau(g)$ defined by the product of matrices on the variables becomes a representation. Similarly, the action of the Lie group $G = O(n)$ on the linear space

$$V = \mathcal{H}_k(\mathbb{C}^n) := \left\{ f(x) \in \mathcal{P}_k(\mathbb{C}^n) \,\middle|\, \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = 0 \right\}$$

(the space of homogeneous harmonic polynomials of n variables, degree k), with $\tau(g)$ defined similarly also becomes a representation. The representation of $U(n)$ on $\mathcal{P}_k(\mathbb{C}^n)$ and that of $O(n)$ on $\mathcal{H}_k(\mathbb{C}^n)$ are examples of irreducible representations. "Irreducible" means that the representation can no longer be decomposed, or there are no linear subspaces $W \subset V$ satisfying $\tau(G)W \subset W$ other than $\{0\}$ and V.

3. Irreducible decompositions of representations—Generalization of Fourier analysis

One of the most fundamental problems in representation theory is to decompose a given representation into a sum of irreducible representations. The irreducible decomposition of the function space (e.g. $L^2(X)$) on X with an action of G is useful for understanding X. For example, $G = O(n)$ acts on the $n-1$-dimensional sphere $S^{n-1} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\}$ by the usual rotations and acts on the function space $L^2(S^{n-1})$ linearly. The space of the restriction of homogeneous harmonic polynomials of degree k on the sphere S^{n-1} (spherical harmonic functions, denoted as $\mathcal{H}_k(\mathbb{C}^n)|_{S^{n-1}} =: \mathcal{H}_k(S^{n-1})$) is then preserved by this action, and becomes a sub-representation. Namely,

If $f(x) \in \mathcal{H}_k(S^{n-1})$, then $\tau(g)f(x) \in \mathcal{H}_k(S^{n-1})$ for all $g \in O(n)$.

Moreover, each $f(x) \in L^2(S^{n-1})$ is expressed uniquely by the form

$$f(x) = \sum_{k=0}^{\infty} f_k(x), \quad f_k(x) \in \mathcal{H}_k(S^{n-1}).$$

Hence, $L^2(S^{n-1})$ is decomposed into the direct sum

$$L^2(S^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^{n-1}).$$

Since each $\mathcal{H}_k(S^{n-1})$ is irreducible, this gives the irreducible decomposition. When $n = 2$, by the coordinate $(\cos\theta, \sin\theta)$ of S^1, we have

$$\begin{aligned} \mathcal{H}_k(S^1) &= \mathbb{C}e^{ik\theta} + \mathbb{C}e^{-ik\theta} = \mathbb{C}\cos k\theta + \mathbb{C}\sin k\theta \\ &= \{ae^{ik\theta} + a'e^{-ik\theta} = (a + a')\cos k\theta + i(a - a')\sin k\theta \mid a, a' \in \mathbb{C}\}, \end{aligned}$$

and the above decomposition coincides with the

Fourier series expansion

$$f(\cos\theta, \sin\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$
$$= a_0 + \sum_{k=1}^{\infty}(b_k \cos k\theta + c_k \sin k\theta)$$

($b_k = a_k + a_{-k}$, $c_k = i(a_k - a_{-k})$). Similarly, the (inverse) Fourier transform

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\, d\xi$$

is regarded as the decomposition of the function space $L^2(\mathbb{R})$ on the real line $\mathbb{R}$ into the sum of one-dimensional sub-representations $\mathbb{C}e^{ix\xi}$ as the representation of the additive group $\mathbb{R}$,

$$L^2(\mathbb{R}) = \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{ix\xi}\, d\xi.$$

Note that this is a sum of uncountably many spaces and called the direct integral instead of the direct sum. Needless to say, the Fourier analysis is important in many fields such as signal processing, and the theory of spherical harmonics is important in the treatment of rotation-invariant systems in quantum physics. Irreducible decompositions of general representations are regarded as generalizations of these important theories.

Among irreducible decompositions, the decomposition of a representation of G under its subgroup $G' \subset G$ is called the branching law. For example, we consider the restriction of the representation $\mathcal{P}_k(\mathbb{C}^n)$ of $G = U(n)$ to the subgroup

$$G' = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \middle| \, g \in U(n-1) \right\} \simeq U(n-1).$$

Then as a representation of $U(n)$, $\mathcal{P}_k(\mathbb{C}^n)$ is irreducible but as a representation of $U(n-1)$, the subspaces

$$\mathcal{P}_m(\mathbb{C}^{n-1})x_n^{k-m} := \{f(x_1, \ldots, x_{n-1})x_n^{k-m} \mid f(x_1, \ldots, x_{n-1}) \in \mathcal{P}_m(\mathbb{C}^{n-1})\} \subset \mathcal{P}_k(\mathbb{C}^n)$$

($m = 0, \ldots, k$) are clearly sub-representations. Therefore, we find that the irreducible decomposition (branching law) of $\mathcal{P}_k(\mathbb{C}^n)$ under $U(n-1)$ is given by

$$\mathcal{P}_k(\mathbb{C}^n)|_{U(n-1)} = \bigoplus_{m=0}^{k} \mathcal{P}_m(\mathbb{C}^{n-1})x_n^{k-m}.$$

Similarly, it is known that the irreducible decomposition of $\mathcal{P}_k(\mathbb{C}^n)$ under $G' = O(n)$ is given by

$$\mathcal{P}_k(\mathbb{C}^n)|_{O(n)} = \bigoplus_{m=0}^{[k/2]} \mathcal{H}_{k-2m}(\mathbb{C}^n)\|x\|^{2m}$$

(where $\|x\|^2 := \sum_{i=1}^{n} x_i^2$). As a more non-trivial example, we consider the branching law of the representation $\mathcal{H}_k(\mathbb{C}^n)$ of $G = O(n)$ under the subgroup $G' \simeq O(n-1)$. For $m = 0, 1, \ldots, k$, let $\tilde{P}_k^m(y)$ be a polynomial of degree at most $k - m$ satisfying $\tilde{P}_k^m(-y) = (-1)^{k-m}\tilde{P}_k^m(y)$, and we consider the linear space

$$W_m := \left\{ \|x\|^{k-m}\tilde{P}_k^m\left(\frac{x_n}{\|x\|}\right) f(x_1, \ldots, x_{n-1}) \middle| f(x_1, \ldots, x_{n-1}) \in \mathcal{H}_m(\mathbb{C}^{n-1}) \right\} \subset \mathcal{P}_k(\mathbb{C}^n).$$

This is then an irreducible representation of $O(n-1)$, and if we suitably choose $\tilde{P}_k^m(y)$, then we can make $W_m \subset \mathcal{H}_k(\mathbb{C}^n)$. In this situation, $\mathcal{H}_k(\mathbb{C}^n)$ is irreducibly decomposed under $O(n-1)$ as

$$\mathcal{H}_k(\mathbb{C}^n)|_{O(n-1)} = \bigoplus_{m=0}^{k} W_m \simeq \bigoplus_{m=0}^{k} \mathcal{H}_m(\mathbb{C}^{n-1}).$$

The polynomial $\tilde{P}_k^m(y)$ is obtained by solving a differential equation concerning the Laplacian and given explicitly by $\tilde{P}_k^m(y) = C_{k-m}^{(m-1+n/2)}(y)$ using the Gegenbauer polynomial $C_k^{(\alpha)}(y)$. When $n = 3$, this is given by a constant multiple of the associated Legendre polynomials. With these polynomials, we can explicitly construct a basis of $\mathcal{H}_k(\mathbb{C}^n)$ inductively on n (**Fig. 2**: $n = 3$).

4. Representations of Lie algebras—Linear approximation of those of Lie groups

Lie groups are generally not linear spaces, and it is not easy to treat their representations directly. To overcome this non-linearity, we consider the Lie algebras associated with the Lie groups instead. For $X \in M(n, \mathbb{C})$ (an $n \times n$ matrix with complex entries) and $t \in \mathbb{R}$, we consider the exponential function $\exp(tX) := \sum_{j=0}^{\infty}(tX)^j/j!$. This satisfies the usual law of exponents $\exp((s + t)X) = \exp(sX)\exp(tX)$ and $\frac{d}{dt}\exp(tX)|_{t=0} = X$. By using this, we define the Lie algebra $Lie(G) \subset M(n, \mathbb{C})$ associated with the Lie group $G \subset GL(n, \mathbb{C})$ by

$$Lie(G) := \{X \in M(n, \mathbb{C}) \mid \exp(tX) \in G \text{ holds for all } t \in \mathbb{R}\}.$$

This then becomes a linear space, and $[X, Y] := XY - YX \in Lie(G)$ holds for all $X, Y \in Lie(G)$. Next, for a finite-dimensional representation (τ, V) of G, we define the representation $(d\tau, V)$ of the Lie algebra $Lie(G)$ by

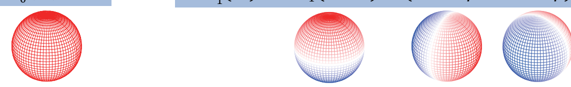
$$d\tau(X)v := \frac{d}{dt}\tau(\exp(tX))v|_{t=0}$$
$$(X \in Lie(G),\ v \in V).$$

Then $d\tau(aX + bY) = ad\tau(X) + bd\tau(Y)$ and $d\tau([X, Y]) = d\tau(X)d\tau(Y) - d\tau(Y)d\tau(X)$ hold for all $X, Y \in Lie(G)$, $a, b \in \mathbb{R}$. This representation $(d\tau, V)$ preserves most information on the original representation (τ, V). For

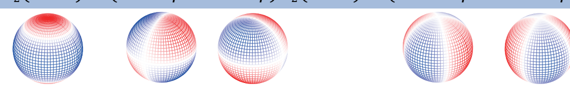
$$\mathcal{H}_k(S^2) = \bigoplus_{m=0}^{k} \tilde{P}_k^m(\cos\theta)\,\mathcal{H}_m(S^1) = \bigoplus_{m=0}^{k} (\mathbb{C}\cos m\phi + \mathbb{C}\sin m\phi)\,P_k^m(\cos\theta),$$

$$P_k^m(\cos\theta) = \sin^m\theta\,\tilde{P}_k^m(\cos\theta) = \frac{\sin^m\theta}{2^k}\sum_{j=0}^{[(k-m)/2]} \frac{(-1)^j\,(2k-2j)!}{j!\,(k-j)!\,(k-2j-m)!}\,(\cos\theta)^{k-2j-m}$$

$\mathcal{H}_0(S^2) = \mathbb{C}$ $\quad$ $\mathcal{H}_1(S^2) = \mathbb{C}P_1^0(\cos\theta) + (\mathbb{C}\cos\phi + \mathbb{C}\sin\phi)P_1^1(\cos\theta)$



$\mathcal{H}_2(S^2) = \mathbb{C}P_2^0(\cos\theta) + (\mathbb{C}\cos\phi + \mathbb{C}\sin\phi)P_2^1(\cos\theta) + (\mathbb{C}\cos 2\phi + \mathbb{C}\sin 2\phi)\,P_2^2(\cos\theta)$



$\mathcal{H}_3(S^2) = \mathbb{C}P_3^0(\cos\theta) + (\mathbb{C}\cos\phi + \mathbb{C}\sin\phi)P_3^1(\cos\theta) + (\mathbb{C}\cos 2\phi + \mathbb{C}\sin 2\phi)P_3^2(\cos\theta) + (\mathbb{C}\cos 3\phi + \mathbb{C}\sin 3\phi)P_3^3(\cos\theta)$

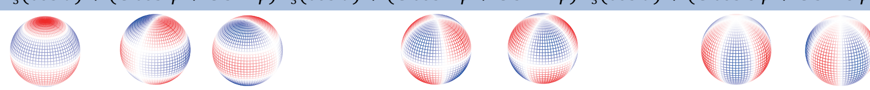


Fig. 2. Express $\mathcal{H}_k(S^2) = \mathcal{H}_k(\mathbb{C}^3)|_{S^2}$ by sum of $\mathcal{H}_m(S^1) = \mathcal{H}_m(\mathbb{C}^2)|_{S^1}$.

$$d\tau(H)f(x,y) = \frac{d}{dt}f\big((x,y)\exp(tH)\big)\Big|_{t=0} = \frac{d}{dt}f\left((x,y)\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\right)\Bigg|_{t=0} = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)f(x,y),$$

$$d\tau(E)f(x,y) = \frac{d}{dt}f\big((x,y)\exp(tE)\big)\Big|_{t=0} = \frac{d}{dt}f\left((x,y)\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\right)\Bigg|_{t=0} = x\frac{\partial}{\partial y}f(x,y),$$

$$d\tau(F)f(x,y) = \frac{d}{dt}f\big((x,y)\exp(tF)\big)\Big|_{t=0} = \frac{d}{dt}f\left((x,y)\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix}\right)\Bigg|_{t=0} = y\frac{\partial}{\partial x}f(x,y).$$

$$\begin{tikzcd}
0 & x^k \arrow[l, "E"'] \arrow[loop above, "H \atop \times k"] \arrow[r, shift right, "F"'] & x^{k-1}y \arrow[l, shift right, "E"'] \arrow[loop above, "H \atop \times k-2"] \arrow[r, shift right, "F"'] & x^{k-2}y^2 \arrow[l, shift right, "E"'] \arrow[loop above, "H \atop \times k-4"] \arrow[r, shift right, "F"'] & \cdots \arrow[l, shift right, "E"'] \arrow[r, shift right, "F"'] & xy^{k-1} \arrow[l, shift right, "E"'] \arrow[loop above, "H \atop \times -k+2"] \arrow[r, shift right, "F"'] & y^k \arrow[l, shift right, "E"'] \arrow[loop above, "H \atop \times -k"] \arrow[r, "F"'] & 0
\end{tikzcd}$$

Fig. 3. Representation of Lie algebra of $SU(2)$ on $\mathcal{P}_k(\mathbb{C}^2)$.

example, if G is connected, then the irreducibility of a finite dimensional representation (τ, V) under G is equivalent to the irreducibility of the differential representation $(d\tau, V)$ under $Lie(G)$.

Let us consider the representation of $G = SU(2) := U(2) \cap SL(2, \mathbb{C})$ on $V = \mathcal{P}_k(\mathbb{C}^2)$ as an example. First, the Lie algebra $Lie(G)$ associated with $G = SU(2)$ and its complexification $Lie(G) \otimes \mathbb{C}$ are given by

$$Lie(G) = \mathfrak{su}(2) := \left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix} \,\middle|\, \begin{matrix} a, b, c \in \mathbb{C}, \\ a = -\bar{a}, b = -\bar{c}\end{matrix}\right\},$$

$$Lie(G) \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}) := \left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix} \,\middle|\, a, b, c \in \mathbb{C}\right\}.$$

We take a basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$ of $\mathfrak{sl}(2, \mathbb{C})$. Then their actions on $\mathcal{P}_k(\mathbb{C}^2)$ are given by

$$d\tau(H)f(x,y) = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)f(x,y),$$

$$d\tau(E)f(x,y) = x\frac{\partial}{\partial y}f(x,y),$$

$$d\tau(F)f(x,y) = y\frac{\partial}{\partial x}f(x,y)$$

(see **Fig. 3** for the intermediate process). In particular, the actions on the basis $\{x^k, x^{k-1}y, x^{k-2}y^2, \ldots, y^k\}$ of $\mathcal{P}_k(\mathbb{C}^2)$ is written as

$$d\tau(H)x^{k-j}y^j = (k-2j)x^{k-j}y^j,$$
$$d\tau(E)x^{k-j}y^j = jx^{k-j+1}y^{j-1},$$
$$d\tau(F)x^{k-j}y^j = (k-j)x^{k-j-1}y^{j+1}.$$

Hence, $d\tau(H)$ has the eigenvalues $\{k, k-2, k-4, ..., -k\}$, $d\tau(E)$ raises the eigenvalue of an eigenvector of $d\tau(H)$ by 2, and $d\tau(F)$ lowers the eigenvalue by 2. This structure is equivalent to those of the spin representations appearing in quantum physics. In fact, a general irreducible representation of $SU(2)$ always has such a structure, especially equivalent to $\mathcal{P}_k(\mathbb{C}^2)$ for a non-negative integer k. As a higher example, we consider the representations of the Lie group $U(n)$. Again, by looking in detail at the action of (the complexification of) the associated Lie algebra for diagonal, upper-triangular, and lower-triangular matrices, we can then show that the set of all irreducible representations of $U(n)$ has a one-to-one correspondence with the set of n-tuples of decreasing integers

$$\{(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

As we have seen above, representations of Lie algebras are helpful for the classification of representations of Lie groups. It is also known that the dimension of each irreducible representation of $U(n)$ is characterized by the number of combinatoric objects called the semistandard Young tableaux.

5. Example of infinite-dimensional representations

Next, we consider an example of infinite-dimensional representations. Let us consider the Lie group $G = SL(2, \mathbb{R})$. Its Lie algebra is then given by

$$Lie(G) = \mathfrak{sl}(2, \mathbb{R}) := \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \, a, b, c \in \mathbb{R}\right\}.$$

We take the basis $H, E, F \in \mathfrak{sl}(2, \mathbb{R})$ as before and define the action of $\mathfrak{sl}(2, \mathbb{R})$ on (some dense subspace of) $V = L^2(\mathbb{R})$ by

$$d\tau(H)f := x\frac{df}{dx} + \frac{1}{2}f, \quad d\tau(E)f = -\frac{i}{2}x^2 f,$$
$$d\tau(F)f = -\frac{i}{2}\frac{d^2 f}{dx^2}.$$

This does not lift to a representation of the Lie group $SL(2, \mathbb{R})$ but lifts to that of the double covering group $\widetilde{SL}(2, \mathbb{R})$ (that is, there exists a representation $(\tau, L^2(\mathbb{R}))$ of $\widetilde{SL}(2, \mathbb{R})$, the differential of which coincides with the above $d\tau$ (on some dense subspace)). The action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp\frac{\pi}{2}(E-F) \in \widetilde{SL}(2, \mathbb{R})$ also coincides with a constant multiple of the Fourier transform.

$$\left(\tau\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right)(x) = \left(\tau\left(\exp\frac{\pi}{2}(E-F)\right)\right)$$
$$= \frac{e^{-\pi i/4}}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\xi)e^{-ix\xi}\, d\xi.$$

This is proved by observing the eigenfunctions of $d\tau(E-F) = -\frac{i}{2}\left(x^2 - \frac{d^2}{dx^2}\right)$ (the quantum harmonic oscillator). While the representation $(\tau, L^2(\mathbb{R}))$ is infinite-dimensional, this is nearly irreducible*2 and called the Weil representation or the metaplectic representation. This construction is generalized to the representation $L^2(\mathbb{R}^n)$ of the larger Lie group $Sp(n, \mathbb{R})$ ($Sp(1, \mathbb{R}) = SL(2, \mathbb{R})$ for $n = 1$ case). By taking the irreducible decomposition of the representation of $Sp(nm, \mathbb{R})$ on $L^2(\mathbb{R}^{nm}) \simeq L^2(M(n, m; \mathbb{R}))$ (the function space on $n \times m$ matrices) under the subgroup $Sp(n, \mathbb{R}) \times O(m) \subset Sp(nm, \mathbb{R})$, we can also obtain various representations of $Sp(n, \mathbb{R})$; thus, this Weil representation is considered quite important in representation theory.

6. Branching law of infinite-dimensional representations

In recent research, I was interested in explicitly determining the branching laws of infinite-dimensional representations (see [2] and references therein). When we restrict a "good" representation (τ, V) of G to a subgroup $G' \subset G$, V is decomposed into a direct sum (or a direct integral) of (in general infinite number of) irreducible representations of G'. Even if we know which representation V' of G' appears abstractly in the decomposition of V, it is generally a difficult problem to determine how V' is included explicitly in V (corresponding to the determination of $\tilde{P}_k^m(y)$ in the example of $\mathcal{H}_k(\mathbb{C}^n)$). I determined the explicit inclusion map (intertwining operator) from V' into V for "good" tuples (G, G', V, V') [3, 4] (**Fig. 4**). There then appear special functions such as hypergeometric functions (and their multivariate generalizations). In the future, I aim to obtain analogous results for more general tuples (G, G', V, V').

*2 The space of all even or odd functions is irreducible, and $L^2(\mathbb{R})$ is a sum of these two irreducible representations.

Definition:

$$Sp(n,\mathbb{R}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2n,\mathbb{R}) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

$$\simeq \left\{ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in GL(2n,\mathbb{C}) \,\middle|\, {}^t g \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

($Sp(1,\mathbb{R}) = SL(2,\mathbb{R}) \simeq SU(1,1)$ for $n=1$),

$D_n := \{x \in M(n,\mathbb{C}) \mid {}^t x = x,\ I - x\bar{x}$ is positive definite$\}$,

$\mathcal{O}(D_n) := \{$holomorphic functions on $D_n\}$.

Define a representation τ_λ of $Sp(n,\mathbb{R})$ on $\mathcal{O}(D_n) = \mathcal{O}_\lambda(D_n)$ by ($\lambda \in \mathbb{C}$)

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}^{-1} \in Sp(n,\mathbb{R}), \qquad \tau(g)\colon \mathcal{O}_\lambda(D_n) \to \mathcal{O}_\lambda(D_n),$$

$$(\tau(g)f)(x) := \det(\bar{b}x + \bar{a})^{-\lambda} f\left((ax+b)(\bar{b}x+\bar{a})^{-1}\right).$$

$$\begin{tikzcd} Sp(2n,\mathbb{R}) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\circlearrowright"] & Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R}) \arrow[d, phantom, "\circlearrowright"] \\ \mathcal{O}_\lambda(D_{2n}) \arrow[r, shift right, "\mathcal{F}^\downarrow"'] & \mathcal{O}_{\lambda+k}(D_n) \otimes \mathcal{O}_{\lambda+k}(D_n) \arrow[l, shift right, "\mathcal{F}^\uparrow"'] \end{tikzcd}$$

If we restrict the representation $\mathcal{O}_\lambda(D_{2n})$ of $Sp(2n,\mathbb{R})$ to the subgroup $Sp(n,\mathbb{R}) \times Sp(n,\mathbb{R})$,
then those equivalent to $\mathcal{O}_{\lambda+k}(D_n) \otimes \mathcal{O}_{\lambda+k}(D_n)$ ($k \in \mathbb{Z}_{\geq 0}$) appear in the decomposition, but it was not known how they were realized.
I proved that their correspondence (intertwining operators) is given by the following differential operators.

$$\mathcal{F}^\uparrow\colon \mathcal{O}_{\lambda+k}(D_n) \otimes \mathcal{O}_{\lambda+k}(D_n) \to \mathcal{O}_\lambda(D_{2n}), \qquad \mathcal{F}^\uparrow f(x) = F^\uparrow\left(x_{12}; \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{22}}\right) f(x_{11}, x_{22}),$$

$$\mathcal{F}^\downarrow\colon \mathcal{O}_\lambda(D_{2n}) \to \mathcal{O}_{\lambda+k}(D_n) \otimes \mathcal{O}_{\lambda+k}(D_n), \qquad \mathcal{F}^\downarrow f(x_{11}, x_{22}) = F^\downarrow\left(\frac{\partial}{\partial x}\right) f(x)\Big|_{x_{12}=0},$$

$$F^\uparrow(x_{12}; y_{11}, y_{22}) := \det(x_{12})^k \sum_{m_1 \geq \cdots \geq m_n \geq 0} \prod_{j=1}^{n} \frac{1}{(\lambda + k - (j-1)/2)_{m_j}} \Phi_{\boldsymbol{m}}(y_{11} x_{12} y_{22} {}^t x_{12})$$

$$F^\downarrow(x) = F^\downarrow\begin{pmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{pmatrix} := \det(x_{12})^k \sum_{m_1 \geq \cdots \geq m_n \geq 0} \prod_{j=1}^{n} \frac{(-(k+j-1)/2)_{m_j} (-(k+j-2)/2)_{m_j}}{(-\lambda - k + n - (j-3)/2)_{m_j}} \Phi_{\boldsymbol{m}}(x_{11} {}^t x_{12}^{-1} x_{22} x_{12}^{-1}).$$

$\Phi_{\boldsymbol{m}}$: homogeneous polynomial of degree $m_1 + \cdots + m_n$, depending only on eigenvalues ($\Phi_m(t) = t^m/m!$ for n=1)

$(\lambda)_m := \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1)$

Fig. 4. Example of my results.

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